

Partial linear spaces and identifying codes

Camino Balbuena¹

Joint work with Araujo, Montejano, Valenzuela

¹Departament de Matemàtica Aplicada III
Universitat Politècnica de Catalunya

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$(1, \leq k)$ -identifying code

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Let $G = (V, E)$ a graph, $C \subseteq V$, $C \neq \emptyset$.

$$I(C; X) = I(X) = \bigcup_{v \in X} N[v] \cap C$$

- ▶ C is said to be a $(1, \leq k)$ -*identifying code* in G if all the sets are different for all $X \subseteq V$ such that $|X| \leq k$.
- ▶ A graph G admits a $(1, \leq k)$ -*identifying code* if there exists such a code C .

A graph formed by a **set of independent edges** cannot admit a $(1, \leq 1)$ -identifying code because

$$N[u] = \{u, v\} = N[v]$$

Characterization

If G admits a $(1, \leq k)$ -identifying code, then $C = V$ is also a $(1, \leq k)$ -identifying code.

A graph G admits a $(1, \leq k)$ -identifying code iff $N[X]$ are mutually different for all $X \subseteq V$ such that $|X| \leq k$.

Known results

Laihonen and Rento [2001]

If G connected admitting a $(1, \leq k)$ -identifying code then
 $\delta(G) \geq k$.

Laihonen [2008] for $k \geq 2$: A k -regular graph of girth

- $g \geq 7$ admits a $(1, \leq k)$ -identifying code.
- $g \geq 5$ admits a $(1, \leq k - 1)$ -identifying code.

Objective

Characterizing bipartite $(k; g)$ -graphs for $g \geq 6$ admitting $(1, \leq k)$ -identifying codes.

Partial linear space

Let (P, \mathcal{L}, I) be such that P **points** and \mathcal{L} **lines**: $L \in \mathcal{L}$ iff $L \subset P$.
The triple (P, \mathcal{L}, I) is said to be a *partial linear space* iff

- $|L| \geq 2$ for all $L \in \mathcal{L}$;
- $|L \cap L'| \leq 1$ for all $L, L' \in \mathcal{L}, L \neq L'$.

(P, \mathcal{L}, I) is *k-regular* if all the lines have k -points.

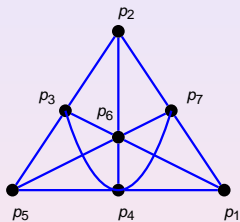
Incidence graph of (P, \mathcal{L}, I)

Bipartite graph B such that

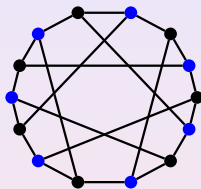
- $V(B) = P \cup \mathcal{L}$;
- $E(B) = I$, that is, $p \sim L$ iff $p \in L$.

girth $g(B) \geq 6$.

Fano Plane



Heawood Graph



Definition

Notation

A partial linear space (P, \mathcal{L}, I) and $X \subseteq P \cup \mathcal{L}$.

$$(X)_I = \cup_{v \in X} \{z : zIv\}.$$

$$[X] = (X)_I \cup X$$

Definition

A k -regular partial linear space (P, \mathcal{L}, I) admits a $(1, \leq k)$ -identifying code iff the sets $[X]$ are mutually distinct for all $X \subseteq P \cup \mathcal{L}$ such that $|X| \leq k$.

Corollary

A k -regular partial linear space (P, \mathcal{L}, I) admits a $(1, \leq k - 1)$ -identifying code for $k \geq 2$.

Theorem

A k -regular partial linear space (P, \mathcal{L}, I) admits a $(1, \leq k)$ -identifying code iff

- (i) for every two collinear $u, p \in P$ there exists $z \in P$ such that

$$|(u)_I \cap (z)_I| + |(p)_I \cap (z)_I| = 1;$$

- (ii) for every two collinear lines $L, M \in \mathcal{L}$ there exists $\Delta \in \mathcal{L}$ such that

$$|(L)_I \cap (\Delta)_I| + |(M)_I \cap (\Delta)_I| = 1.$$

Theorem

A k -regular bipartite graph B of girth at least 6 admits a $(1, \leq k)$ -identifying code iff for every two vertices $u, v \in V(B)$ such that $|N(u) \cap N(v)| = 1$, there exists $z \in V(B)$ such that

$$|N(u) \cap N(z)| + |N(v) \cap N(z)| = 1.$$

Projective planes

A projective plane of order $k - 1$ is a k -regular partial linear space such that any two distinct points are collinear and any two distinct lines are concurrent.

A $(k, 6)$ -cage is the incidence graph of a projective plane of order $k - 1$.

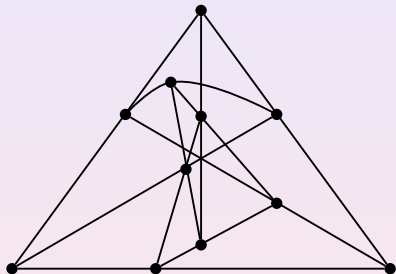
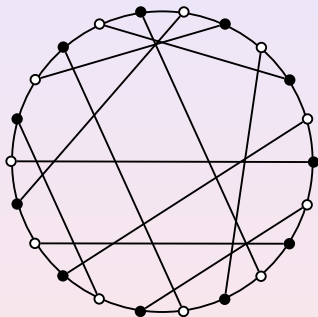
Corollary

A projective plane of order $k - 1$ does not admit a $(1, \leq k)$ -identifying code.

Equivalently:

A $(k, 6)$ -cage does not admit a $(1, \leq k)$ -identifying code.

A (3,6)-bipartite graph on 22 vertices without $(1, \leq 3)$ codes and its corresponding partial linear space.



$(q, 6)$ -graphs

Let $q \geq 2$ be a prime power and Γ_q the $(q + 1, 6)$ -cage.

Suppose $A \subset V(\Gamma_q)$ such that for all $u \in V \setminus A$, $|N(u) \cap A| = 1$.

Then $\Gamma_q - A$ is a $(q, 6)$ -graph on $2(q^2 + q + 1) - |A|$ vertices.

- ▶ Geometrical language: A is called *good structure*.
- ▶ Graphical language: A is called a *perfect dominating set*.

Perfect dominating set in a $\Gamma_q = (q + 1, 6)$ -cage of $2(q + 1)$ vertices:

Let $p, L \in V(\Gamma_q)$ be such that $p \notin L$.

$$N[p] \cup N[L]$$

is a perfect dominating set in Γ_q .

Then $\Gamma_q - (N[p] \cup N[L])$ is a $(q, 6)$ -bipartite graph on $2q^2 - 1$ vertices.

These graphs admit $(1, \leq q)$ -identifying codes.

A 3-regular partial linear space of 8 points and 8 lines admitting $(1, \leq 3)$ -identifying code and its corresponding $(3,6)$ -bipartite graph on 16 vertices.

